

# MULTILINEAR DYADIC OPERATORS AND THEIR COMMUTATORS

ISHWARI KUNWAR

**ABSTRACT.** We introduce multilinear analogues of dyadic paraproduct operators and Haar Multipliers, and study boundedness properties of these operators and their commutators. We also characterize dyadic  $BMO$  functions via boundedness of certain paraproducts and also via boundedness of the commutators of multilinear Haar Multipliers and paraproduct operators.

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## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Dyadic operators have attracted a lot of attention in the recent years. The proof of so-called  $A_2$  theorem (see [7]) consisted in representing a general Calderón-Zygmund operator as an average of dyadic shifts, and then verifying some testing conditions for those simpler dyadic operators. It seems reasonable to believe that, taking a similar approach, general multilinear Calderón-Zygmund operators can be studied by studying multilinear dyadic operators. Regardless of this possibility, multilinear dyadic operators in their own right are an important class of objects in Harmonic Analysis. Statements regarding those operators can be translated into the non-dyadic world, and are sometimes simpler to prove.

In this paper we introduce multilinear analogues of dyadic operators such as paraproducts and Haar multipliers, and study their boundedness properties. Corresponding theory of linear dyadic operators, which we will be using very often, can be found in [11]. In [1], the authors have studied boundedness properties of bilinear paraproducts defined in terms of so-called “smooth molecules”. The paraproduct operators we study are more general multilinear operators, but defined in terms of indicators and Haar functions of dyadic intervals. In [3] Coifman, Rochberg and Weiss proved that the commutator of a  $BMO$  function with a singular integral operator is bounded in  $L^p$ ,  $1 < p < \infty$ . The necessity of  $BMO$  condition for the boundedness of the commutator was also established for certain singular integral operators, such as the Hilbert transform. S. Janson [8] later studied its analogue for linear

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martingale transforms. In this paper we study commutators of multilinear dyadic operators, and characterize dyadic  $BMO$  functions via boundedness of these commutators. For the corresponding theory for general multilinear Calderón-Zygmund operators we refer to [5] and [10].

We organize the paper as follows:

In section 2, we present an overview of some of the main tools we will be using in this paper. These include: the Haar system, linear Haar multipliers, dyadic maximal/square functions, linear/bilinear paraproduct operators and the space of dyadic  $BMO$  functions. For more details we refer to [11].

In section 3, we obtain a decomposition of the pointwise product of  $m$  functions,  $m \geq 2$ , which generalizes the paraproduct decomposition of two functions. On the basis of this decomposition we define multilinear paraproducts and investigate their boundedness properties as operators on products of Lebesgue spaces. We also define multilinear analogue of the linear paraproduct operator  $\pi_b$ , and characterize dyadic  $BMO$  functions via boundedness of certain multilinear paraproduct operators.

In section 4, we define multilinear Haar multipliers in a way consistent with the definition of linear Haar multipliers and multilinear paraproducts, and then investigate their boundedness properties. We also study boundedness properties of their commutators with dyadic  $BMO$  functions, and provide a characterization of dyadic  $BMO$  functions via the boundedness of those multilinear commutators. In particular, we show that the commutators of the multilinear paraproducts with a function  $b$  are bounded if and only if  $b$  is a dyadic  $BMO$  function.

Our main results involve the following operators:

- $P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \left( \prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{\sigma(\vec{\alpha})}, \quad \vec{\alpha} \in \{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}.$
- $\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \left( \prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{1+\sigma(\vec{\alpha})}, \quad \vec{\alpha} \in \{0, 1\}^m.$
- $T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) := \sum_{I \in \mathcal{D}} \epsilon_I \left( \prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{\sigma(\vec{\alpha})},$   
 $\vec{\alpha} \in \{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}, \epsilon = \{\epsilon_I\}_{I \in \mathcal{D}} \text{ bounded.}$
- $[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)(x) := b(x)T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)(x) - T_\epsilon^{\vec{\alpha}}(f_1, \dots, b f_i, \dots, f_m)(x),$   
 $1 \leq i \leq m, \vec{\alpha} \in \{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}, \epsilon = \{\epsilon_I\}_{I \in \mathcal{D}} \text{ bounded and } b \in BMO^d.$

In the above definitions,  $\mathcal{D} := \{[m2^{-k}, (m+1)2^{-k}) : m, k \in \mathbb{Z}\}$  is the standard dyadic grid on  $\mathbb{R}$  and  $h_I$ 's are the Haar functions defined by  $h_I = \frac{1}{|I|^{1/2}} (1_{I_+} - 1_{I_-})$ , where  $I_-$  and  $I_+$

are the left and right halves of  $I$ . With  $\langle \cdot, \cdot \rangle$  denoting the standard inner product in  $L^2(\mathbb{R})$ ,  $f_i(I, 0) := \langle f_i, h_I \rangle$  and  $f_i(I, 1) := \langle f_i, h_I^2 \rangle = \frac{1}{|I|} \int_I f_i$ , the average of  $f_i$  over  $I$ . The Haar coefficient  $\langle f_i, h_I \rangle$  is sometimes denoted by  $\widehat{f_i}(I)$  and the average of  $f_i$  over  $I$  by  $\langle f_i \rangle_I$ . For  $\vec{\alpha} \in \{0, 1\}^m$ ,  $\sigma(\vec{\alpha})$  denotes the number of 0 components in  $\vec{\alpha}$ . For convenience, we will denote the set  $\{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}$  by  $U_m$ .

In the following main results  $L^p$  stands for the Lebesgue space  $L^p(\mathbb{R}) := \{f : \|f\|_p < \infty\}$  with  $\|f\|_p = \|f\|_{L^p} := \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}$ . The Weak  $L^p$  space, also denoted by  $L^{p,\infty}$ , is the space of all functions  $f$  such that

$$\|f\|_{L^{p,\infty}(\mathbb{R})} := \sup_{t>0} t |\{x \in \mathbb{R} : f(x) > t\}|^{1/p} < \infty.$$

Moreover,  $\|b\|_{BMO^d} := \sup_{I \in \mathcal{D}} \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I| dx < \infty$ , is the dyadic  $BMO$  norm of  $b$ .

We now state our main results:

**Theorem:** Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$  and  $1 < p_1, p_2, \dots, p_m < \infty$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ .

Then

- (a) For  $\vec{\alpha} \neq (1, 1, \dots, 1)$ ,  $\|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \lesssim \prod_{j=1}^m \|f_j\|_{p_j}$ .
- (b) For  $\sigma(\vec{\alpha}) \leq 1$ ,  $\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}$ , if and only if  $b \in BMO^d$ .
- (c) For  $\sigma(\vec{\alpha}) > 1$ ,  $\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \leq C_b \prod_{j=1}^m \|f_j\|_{p_j}$ , if and only if  $\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$ .

In each of the above cases, the paraproducts are weakly bounded if  $1 \leq p_1, p_2, \dots, p_m < \infty$ .

**Theorem:** Let  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$  be a given sequence and let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ . Let  $1 < p_1, p_2, \dots, p_m < \infty$  with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

Then  $T_{\epsilon}^{\vec{\alpha}}$  is bounded from  $L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}$  to  $L^r$  if and only if  $\|\epsilon\|_{\infty} := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$ .

Moreover,  $T_{\epsilon}^{\vec{\alpha}}$  has the corresponding weak-type boundedness if  $1 \leq p_1, p_2, \dots, p_m < \infty$ .

**Theorem:** Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ ,  $1 \leq i \leq m$ , and  $1 < p_1, p_2, \dots, p_m, r < \infty$  with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

Suppose  $b \in L^p$  for some  $p \in (1, \infty)$ . Then the following two statements are equivalent.

(a)  $b \in BMO^d$ .

(b)  $[b, T_\epsilon^\alpha]_i : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \rightarrow L^r$  is bounded for every bounded sequence  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$ .

In particular,  $b \in BMO^d$  if and only if  $[b, P^\alpha]_i : L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \rightarrow L^r$  is bounded.

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## 2. NOTATION AND PRELIMINARIES

**2.1. The Haar System and the Haar multipliers:** Let  $\mathcal{D}$  denote the standard dyadic grid on  $\mathbb{R}$ ,

$$\mathcal{D} = \{[m2^{-k}, (m+1)2^{-k}) : m, k \in \mathbb{Z}\}.$$

Associated to each dyadic interval  $I$  there is a Haar function  $h_I$  defined by

$$h_I(x) = \frac{1}{|I|^{1/2}} (1_{I_+} - 1_{I_-}),$$

where  $I_-$  and  $I_+$  are the left and right halves of  $I$ .

The collection of all Haar functions  $\{h_I : I \in \mathcal{D}\}$  is an orthonormal basis of  $L^2(\mathbb{R})$ , and an unconditional basis of  $L^p$  for  $1 < p < \infty$ . In fact, if a sequence  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$  is bounded, the operator  $T_\epsilon$  defined by

$$T_\epsilon f(x) = \sum_{I \in \mathcal{D}} \epsilon_I \langle f, h_I \rangle h_I$$

is bounded in  $L^p$  for all  $1 < p < \infty$ . The converse also holds. The operator  $T_\epsilon$  is called the Haar multiplier with symbol  $\epsilon$ .

**2.2. The dyadic maximal function:** Given a function  $f$ , the dyadic Hardy-Littlewood maximal function  $M^d f$  is defined by

$$M^d f(x) := \sup_{x \in I \in \mathcal{D}} \frac{1}{|I|} \int_I |f(t)| dt.$$

For the convenience of notation, we will just write  $M$  to denote the dyadic maximal operator. Clearly,  $M$  is bounded on  $L^\infty$ . It is well-known that  $M$  is of weak type  $(1, 1)$  and strong type  $(p, p)$  for all  $1 < p < \infty$ .

**2.3. The dyadic square function:** The dyadic Littlewood-Paley square function of a function  $f$  is defined by

$$Sf(x) := \left( \sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I|} 1_I(x) \right)^{1/2}.$$

For  $f \in L^p$  with  $1 < p < \infty$ , we have  $\|Sf\|_p \approx \|f\|_p$  with equality when  $p = 2$ .

**2.4. BMO Space.** A locally integrable function  $b$  is said to be of bounded mean oscillation if

$$\|b\|_{BMO} := \sup_I \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I| dx < \infty,$$

where the supremum is taken over all intervals in  $\mathbb{R}$ . The space of all functions of bounded mean oscillation is denoted by  $BMO$ .

If we take the supremum over all dyadic intervals in  $\mathbb{R}$ , we get a larger space of dyadic BMO functions which we denote by  $BMO^d$ .

For  $0 < r < \infty$ , define

$$BMO_r = \{b \in L_{loc}^r(\mathbb{R}) : \|b\|_{BMO_r} < \infty\},$$

$$\text{where, } \|b\|_{BMO_r} := \left( \sup_I \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I|^r dx \right)^{1/r}.$$

For any  $0 < r < \infty$ , the norms  $\|b\|_{BMO_r}$  and  $\|b\|_{BMO}$  are equivalent. The equivalence of norms for  $r > 1$  is well-known and follows from John-Nirenberg's lemma (see [9]), while the equivalence for  $0 < r < 1$  has been proved by Hanks in [6]. (See also [12], page 179.)

For  $r = 2$ , it follows from the orthogonality of Haar system that

$$\|b\|_{BMO_2^d} = \left( \sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \subseteq I} |\widehat{b}(J)|^2 \right)^{1/2}.$$

**2.5. The linear/ bilinear paraproducts:** Given two functions  $f_1$  and  $f_2$ , the point-wise product  $f_1 f_2$  can be decomposed into the sum of bilinear paraproducts:

$$f_1 f_2 = P^{(0,0)}(f_1, f_2) + P^{(0,1)}(f_1, f_2) + P^{(1,0)}(f_1, f_2),$$

where for  $\vec{\alpha} = (\alpha_1, \alpha_2) \in \{0, 1\}^2$ ,

$$P^{\vec{\alpha}}(f_1, f_2) = \sum_{I \in \mathcal{D}} f_1(I, \alpha_1) f_2(I, \alpha_2) h_I^{\sigma(\vec{\alpha})}$$

with  $f_i(I, 0) = \langle f_i, h_I \rangle$ ,  $f_i(I, 1) = \langle f_i \rangle_I$ ,  $\sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$ , and  $h_I^{\sigma(\vec{\alpha})}$  being the point-wise product  $h_I h_I \dots h_I$  of  $\sigma(\vec{\alpha})$  factors.

The paraproduct  $P^{(0,1)}(f_1, f_2)$  is also denoted by  $\pi_{f_1}(f_2)$ , i.e.,

$$\pi_{f_1}(f_2) = \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I h_I.$$

Observe that

$$\langle \pi_{f_1}(f_2), g \rangle = \left\langle \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I h_I, g \right\rangle = \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I \langle g, h_I \rangle$$

which is equal to

$$\begin{aligned}
\langle f_2, P^{(0,0)}(f_1, g) \rangle &= \left\langle f_2, \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle g, h_I \rangle h_I^2 \right\rangle \\
&= \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle g, h_I \rangle \langle f_2, h_I^2 \rangle \\
&= \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I \langle g, h_I \rangle.
\end{aligned}$$

This shows that  $\pi_{f_1}^* = P^{(0,0)}(f_1, \cdot) = P^{(0,0)}(\cdot, f_1)$ .

The ordinary multiplication operator  $M_b : f \rightarrow bf$  can therefore be given by:

$$M_b(f) = bf = P^{(0,0)}(b, f) + P^{(0,1)}(b, f) + P^{(1,0)}(b, f) = \pi_b^*(f) + \pi_b(f) + \pi_f(b).$$

The function  $b$  is required to be in  $L^\infty$  for the boundedness of  $M_b$  in  $L^p$ . However, the paraproduct operator  $\pi_b$  is bounded in  $L^p$  for every  $1 < p < \infty$  if  $b \in BMO^d$ . Note that  $BMO^d$  properly contains  $L^\infty$ . Detailed information on the operator  $\pi_b$  can be found in [11] or [2].

**2.6. Commutators of Haar multipliers:** The commutator of  $T_\epsilon$  with a locally integrable function  $b$  is defined by

$$[b, T_\epsilon](f)(x) := T_\epsilon(bf)(x) - M_b(T_\epsilon(f))(x).$$

It is well-known that for a bounded sequence  $\epsilon$  and  $1 < p < \infty$ , the commutator  $[b, T_\epsilon]$  is bounded in  $L^p$  for all  $p \in (1, \infty)$  if  $b \in BMO^d$ .

These commutators have been studied in [13] in non-homogeneous martingale settings.

### 3. MULTILINEAR DYADIC PARAPRODUCTS

**3.1. Decomposition of pointwise product  $\prod_{j=1}^m f_j$ .** In this sub-section we obtain a de-

composition of pointwise product  $\prod_{j=1}^m f_j$  of  $m$  functions that is analogous to the following paraproduct decomposition :

$$f_1 f_2 = P^{(0,0)}(f_1, f_2) + P^{(0,1)}(f_1, f_2) + P^{(1,0)}(f_1, f_2).$$

The decomposition of  $\prod_{j=1}^m f_j$  will be the basis for defining *multi-linear paraproducts* and *m-linear Haar multipliers*, and will also be very useful in proving boundedness properties of multilinear commutators.

We first introduce the following notation:

$$\bullet \quad f(I, 0) := \widehat{f}(I) = \langle f, h_I \rangle = \int_{\mathbb{R}} f(x) h_I(x) dx.$$

- $f(I, 1) := \langle f \rangle_I = \frac{1}{|I|} \int_I f(x) dx.$
- $U_m := \{(\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m : (\alpha_1, \alpha_2, \dots, \alpha_m) \neq (1, 1, \dots, 1)\}.$
- $\sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$  for  $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in \{0, 1\}^m.$
- $(\vec{\alpha}, i) = (\alpha_1, \dots, \alpha_m, i), (i, \vec{\alpha}) = (i, \alpha_1, \dots, \alpha_m)$  for  $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in \{0, 1\}^m.$
- $P_I^{\vec{\alpha}}(f_1, \dots, f_m) = \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$  for  $\vec{\alpha} \in U_m$  and  $I \in \mathcal{D}.$
- $P^{\vec{\alpha}}(f_1, \dots, f_m) = \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, \dots, f_m) = \sum_{I \in \mathcal{D}} \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$  for  $\vec{\alpha} \in U_m.$

With this notation, the paraproduct decomposition of  $f_1 f_2$  takes the following form:

$$f_1 f_2 = P^{(0,0)}(f_1, f_2) + P^{(0,1)}(f_1, f_2) + P^{(1,0)}(f_1, f_2) = \sum_{\vec{\alpha} \in U_2} P^{\vec{\alpha}}(f_1, f_2).$$

Note that

$$(3.1) \quad U_m = \{(\alpha, 1) : \vec{\alpha} \in U_{m-1}\} \cup \{(\vec{\alpha}, 0) : \vec{\alpha} \in U_{m-1}\} \cup \{(1, \dots, 1, 0)\}.$$

To obtain an analogous decomposition of  $\prod_{j=1}^m f_j$ , we need the following crucial lemma:

**Lemma 3.1.** *Given  $m \geq 2$  and functions  $f_1, f_2, \dots, f_m$ , with  $f_i \in L^{p_i}, 1 < p_i < \infty$ , we have*

$$\prod_{j=1}^m \langle f_j \rangle_J 1_J = \sum_{\vec{\alpha} \in U_m} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_m) 1_J,$$

for all  $J \in \mathcal{D}.$

*Proof.* We prove the lemma by induction on  $m.$

First assume that  $m = 2.$  We want to prove the following:

$$\begin{aligned} (3.2) \quad \langle f_1 \rangle_J \langle f_2 \rangle_J 1_J &= \sum_{\vec{\alpha} \in U_2} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2) 1_J \\ &= \left( \sum_{J \subsetneq I} P_I^{(0,1)}(f_1, f_2) + \sum_{J \subsetneq I} P_I^{(1,0)}(f_1, f_2) + \sum_{J \subsetneq I} P_I^{(0,0)}(f_1, f_2) \right) 1_J \\ &= \left( \sum_{J \subsetneq I} \widehat{f_1}(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \langle f_1 \rangle_I \widehat{f_2}(I) h_I + \sum_{J \subsetneq I} \widehat{f_1}(I) \widehat{f_2}(I) h_I^2 \right) 1_J. \end{aligned}$$

We have,

$$\begin{aligned}
& \langle f_1 \rangle_J \langle f_2 \rangle_J \mathbf{1}_J \\
&= \left( \sum_{J \subsetneq I} \widehat{f}_1(I) h_I \right) \left( \sum_{J \subsetneq K} \widehat{f}_2(K) h_K \right) \mathbf{1}_J \\
&= \sum_{J \subsetneq I} \widehat{f}_1(I) h_I \left( \sum_{J \subsetneq K} \widehat{f}_2(K) h_K + \widehat{f}_2(I) h_I + \sum_{J \subsetneq K \subsetneq I} \widehat{f}_2(K) h_K \right) \mathbf{1}_J \\
&= \left\{ \sum_{J \subsetneq I} \widehat{f}_1(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f}_1(I) \widehat{f}_2(I) h_I^2 + \sum_{J \subsetneq I} \widehat{f}_1(I) h_I \left( \sum_{J \subsetneq K \subsetneq I} \widehat{f}_2(K) h_K \right) \right\} \mathbf{1}_J \\
&= \left\{ \sum_{J \subsetneq I} \widehat{f}_1(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f}_1(I) \widehat{f}_2(I) h_I^2 + \sum_{J \subsetneq K} \widehat{f}_2(K) h_K \left( \sum_{K \subsetneq I} \widehat{f}_1(I) h_I \right) \right\} \mathbf{1}_J \\
&= \left\{ \sum_{J \subsetneq I} \widehat{f}_1(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f}_1(I) \widehat{f}_2(I) h_I^2 + \sum_{J \subsetneq K} \widehat{f}_2(K) \langle f_1 \rangle_K h_K \right\} \mathbf{1}_J \\
&= \left\{ \sum_{J \subsetneq I} \widehat{f}_1(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \widehat{f}_1(I) \widehat{f}_2(I) h_I^2 + \sum_{J \subsetneq I} \widehat{f}_2(I) \langle f_1 \rangle_I h_I \right\} \mathbf{1}_J \\
&= \left( \sum_{J \subsetneq I} \widehat{f}_1(I) \langle f_2 \rangle_I h_I + \sum_{J \subsetneq I} \langle f_1 \rangle_I \widehat{f}_2(I) h_I + \sum_{J \subsetneq I} \widehat{f}_1(I) \widehat{f}_2(I) h_I^2 \right) \mathbf{1}_J.
\end{aligned}$$

Now assume  $m > 2$  and that

$$\prod_{j=1}^{m-1} \langle f_j \rangle_J \mathbf{1}_J = \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \mathbf{1}_J.$$

Then,

$$\begin{aligned}
& \prod_{j=1}^m \langle f_j \rangle_J \mathbf{1}_J \\
&= \left( \prod_{j=1}^{m-1} \langle f_j \rangle_J \mathbf{1}_J \right) \langle f_m \rangle_J \mathbf{1}_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \left( \sum_{J \subsetneq K} \widehat{f}_m(K) h_K \right) \mathbf{1}_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \left( \sum_{I \subsetneq K} \widehat{f}_m(K) h_K + \widehat{f}_m(I) h_I + \sum_{J \subsetneq K \subsetneq I} \widehat{f}_m(K) h_K \right) \mathbf{1}_J
\end{aligned}$$



This gives

$$\begin{aligned}
& \prod_{j=1}^m \langle f_j \rangle_J 1_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \langle f_m \rangle_I 1_J + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \widehat{f_m}(I) h_I 1_J \\
&\quad + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \left( \sum_{J \subsetneq K \subsetneq I} \widehat{f_m}(K) h_K \right) 1_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{(\vec{\alpha}, 1)}(f_1, f_2, \dots, f_m) 1_J + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{(\vec{\alpha}, 0)}(f_1, f_2, \dots, f_m) 1_J \\
&\quad + \sum_{J \subsetneq K} \widehat{f_2}(K) h_K \left( \sum_{\vec{\alpha} \in U_{m-1}} \sum_{K \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \right) 1_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{(\vec{\alpha}, 1)}(f_1, f_2, \dots, f_m) 1_J + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{(\vec{\alpha}, 0)}(f_1, f_2, \dots, f_m) 1_J \\
&\quad + \sum_{J \subsetneq K} \widehat{f_m}(K) h_K \langle f_1 \rangle_K \dots \langle f_{m-1} \rangle_K 1_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{(\vec{\alpha}, 1)}(f_1, f_2, \dots, f_m) 1_J + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{(\vec{\alpha}, 0)}(f_1, f_2, \dots, f_m) 1_J \\
&\quad + \sum_{J \subsetneq I} P_I^{(1, \dots, 1, 0)}(f_1, f_2, \dots, f_m) 1_J \\
&= \sum_{\vec{\alpha} \in U_m} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_m) 1_J.
\end{aligned}$$

The last equality follows from (3.1). □

**Lemma 3.2.** *Given  $m \geq 2$  and functions  $f_1, f_2, \dots, f_m$ , with  $f_i \in L^{p_i}$ ,  $1 < p_i < \infty$ , we have*

$$\prod_{j=1}^m f_j = \sum_{\vec{\alpha} \in U_m} P^{\vec{\alpha}}(f_1, f_2, \dots, f_m).$$

*Proof.* We have already seen that it is true for  $m = 2$ . By induction, assume that

$$\begin{aligned}
\prod_{j=1}^{m-1} f_j &= \sum_{\vec{\alpha} \in U_{m-1}} P^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1})
\end{aligned}$$

Then,

$$\begin{aligned}
\prod_{j=1}^m f_j &= \left( \prod_{j=1}^{m-1} f_j \right) f_m \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \left( \sum_{J \in \mathcal{D}} \widehat{f_m}(J) h_J \right) \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \left( \sum_{I \subsetneq J} \widehat{f_m}(J) h_J + \widehat{f_m}(I) h_I + \sum_{J \subsetneq I} \widehat{f_m}(J) h_J \right) \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \langle f_m \rangle_I + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \widehat{f_m}(I) h_I \\
&\quad + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \left( \sum_{J \subsetneq I} \widehat{f_m}(J) h_J \right) \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{(\vec{\alpha}, 1)}(f_1, f_2, \dots, f_m) + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{(\vec{\alpha}, 0)}(f_1, f_2, \dots, f_m) \\
&\quad + \sum_J \widehat{f_m}(J) h_J \left( \sum_{\vec{\alpha} \in U_{m-1}} \sum_{J \subsetneq I} P_I^{\vec{\alpha}}(f_1, f_2, \dots, f_{m-1}) \right) \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{(\vec{\alpha}, 1)}(f_1, f_2, \dots, f_m) + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{(\vec{\alpha}, 0)}(f_1, f_2, \dots, f_m) \\
&\quad + \sum_J \widehat{f_m}(J) h_J \langle f_1 \rangle_J \dots \langle f_{m-1} \rangle_J \\
&= \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{(\vec{\alpha}, 1)}(f_1, f_2, \dots, f_m) + \sum_{\vec{\alpha} \in U_{m-1}} \sum_{I \in \mathcal{D}} P_I^{(\vec{\alpha}, 0)}(f_1, f_2, \dots, f_m) \\
&\quad + P^{(1, \dots, 1, 0)}(f_1, f_2, \dots, f_m) \\
&= \sum_{\vec{\alpha} \in U_m} P^{\vec{\alpha}}(f_1, f_2, \dots, f_m).
\end{aligned}$$

Here the last equality follows from (3.1).  $\square$

**3.2. Multilinear dyadic paraproducts.** On the basis of the decomposition of pointwise product  $\prod_{j=1}^m f_j$  we now define multi-linear dyadic paraproduct operators, and study their boundedness properties.

**Definition 3.1.** For  $m \geq 2$  and  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$ , we define multi-linear dyadic paraproduct operators by

$$P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$$

where  $f_i(I, 0) = \langle f_i, h_I \rangle$ ,  $f_i(I, 1) = \langle f_i \rangle_I$  and  $\sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$ .

Observe that if  $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_m)$  is some permutation of  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$  and  $(g_1, g_2, \dots, g_m)$  is the corresponding permutation of  $(f_1, f_2, \dots, f_m)$ , then

$$P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = P^{\vec{\beta}}(g_1, g_2, \dots, g_m).$$

Also note that  $P^{(1,0)}$  and  $P^{(0,1)}$  are the standard bilinear paraproduct operators:

$$P^{(0,1)}(f_1, f_2) = \sum_{I \in \mathcal{D}} \langle f_1, h_I \rangle \langle f_2 \rangle_I h_I = P(f_1, f_2)$$

$$P^{(1,0)}(f_1, f_2) = \sum_{I \in \mathcal{D}} \langle f_1 \rangle_I \langle f_2, h_I \rangle h_I = P(f_1, f_2).$$

In terms of paraproducts, the decomposition of point-wise product  $\prod_{j=1}^m f_j$  we obtained in the previous section takes the form

$$\prod_{j=1}^m f_j = \sum_{\substack{\vec{\alpha} \in \{0,1\}^m \\ \vec{\alpha} \neq (1,1,\dots,1)}} P^{\vec{\alpha}}(f_1, f_2, \dots, f_m).$$

**Definition 3.2.** For a given function  $b$  and  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$ , we define the paraproduct operators  $\pi_b^{\vec{\alpha}}$  by

$$\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = P^{(0, \vec{\alpha})}(b, f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \prod_{j=1}^m f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})}$$

where  $(0, \vec{\alpha}) = (0, \alpha_1, \dots, \alpha_m) \in \{0, 1\}^{m+1}$ .

Note that

$$\pi_b^1(f) = P^{(0,1)}(b, f) = \sum_{I \in \mathcal{D}} b(I, 0) f(I, 1) h_I = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f \rangle_I h_I = \pi_b(f).$$

The rest of this section is devoted to the boundedness properties of these multilinear paraproduct operators  $P^{\vec{\alpha}}$  and  $\pi_b^{\vec{\alpha}}$ .

**Lemma 3.3.** Let  $1 < p_1, p_2, \dots, p_m, r < \infty$  and  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ . Then for  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ , the operators  $P^{\vec{\alpha}}$  map  $L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r$  with estimates of the form:

$$\|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \lesssim \prod_{j=1}^m \|f_j\|_{p_j}$$

*Proof.* First we observe that, if  $x \in I \in \mathcal{D}$ , then

$$|\langle f \rangle_I| \leq \langle |f| \rangle_I \leq Mf(x)$$

and that

$$\begin{aligned}
\frac{|\langle f, h_I \rangle|}{\sqrt{|I|}} &= \frac{1}{\sqrt{|I|}} \left| \int_{\mathbb{R}} f h_I \right| \\
&= \frac{1}{|I|} \left| \int_{\mathbb{R}} f 1_{I_+} - \int_{\mathbb{R}} f 1_{I_-} \right| \\
&= \frac{1}{|I|} \left( \int_{I_+} |f| + \int_{I_-} |f| \right) \\
&\leq \frac{1}{|I|} \int_I |f| \\
&\leq Mf(x).
\end{aligned}$$

**Case I:**  $\sigma(\vec{\alpha}) = 1$ .

Let  $\alpha_{j_0} = 0$ . Then

$$\begin{aligned}
P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) &= \sum_{I \in \mathcal{D}} \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= \sum_{I \in \mathcal{D}} \left( \prod_{\substack{j=1 \\ j \neq j_0}}^m \langle f_j \rangle_I \right) \langle f_{j_0}, h_I \rangle h_I.
\end{aligned}$$

Using square function estimates, we obtain

$$\begin{aligned}
\|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r &\lesssim \left\| \left( \sum_{I \in \mathcal{D}} \prod_{\substack{j=1 \\ j \neq j_0}}^m |\langle f_j \rangle_I|^2 |\langle f_{j_0}, h_I \rangle|^2 \frac{1_I}{|I|} \right)^{1/2} \right\|_r \\
&\leq \left\| \left( \prod_{\substack{j=1 \\ j \neq j_0}}^m Mf_j \right) \left( \sum_{I \in \mathcal{D}} |\langle f_{j_0}, h_I \rangle|^2 \frac{1_I}{|I|} \right)^{1/2} \right\|_r \\
&= \left\| \left( \prod_{\substack{j=1 \\ j \neq j_0}}^m Mf_j \right) (Sf_{j_0}) \right\|_r \\
&\leq \prod_{\substack{j=1 \\ j \neq j_0}}^m \|Mf_j\|_{p_j} \|Sf_{j_0}\|_{j_0} \\
&\lesssim \prod_{j=1}^m \|f_j\|_{p_j},
\end{aligned}$$

where we have used Hölder inequality, and the boundedness of maximal and square function operators to obtain the last two inequalities.

**Case II:**  $\sigma(\vec{\alpha}) > 1$ .

Choose  $j'$  and  $j''$  such that  $\alpha_{j'} = \alpha_{j''} = 0$ . Then

$$\begin{aligned} |P^{(0,0,\dots,0)}(f_1, f_2, \dots, f_m)(x)| &= \left| \sum_{I \in \mathcal{D}} \left( \prod_{j: \alpha_j=1} \langle f_j \rangle_I \right) \left( \prod_{\substack{j: \alpha_j=0 \\ j \neq j', j''}} \frac{\langle f_j, h_I \rangle}{\sqrt{|I|}} \right) \langle f_{j'}, h_I \rangle \langle f_{j''}, h_I \rangle \frac{1_I(x)}{|I|} \right| \\ &\leq \left( \prod_{j: j \neq j', j''} M f_j(x) \right) \left( \sum_{I \in \mathcal{D}} |\langle f_{j'}, h_I \rangle| |\langle f_{j''}, h_I \rangle| \frac{1_I(x)}{|I|} \right). \end{aligned}$$

By Cauchy-Schwarz inequality

$$\begin{aligned} (3.3) \quad & \sum_{I \in \mathcal{D}} |\langle f_{j'}, h_I \rangle| |\langle f_{j''}, h_I \rangle| \frac{1_I(x)}{|I|} \\ & \leq \left( \sum_{I \in \mathcal{D}} |\langle f_{j'}, h_I \rangle|^2 \frac{1_I(x)}{|I|} \right)^{\frac{1}{2}} \left( \sum_{I \in \mathcal{D}} |\langle f_{j''}, h_I \rangle|^2 \frac{1_I(x)}{|I|} \right)^{\frac{1}{2}} \\ & = S f_{j'}(x) S f_{j''}(x). \end{aligned}$$

Therefore,

$$|P^{(0,0,\dots,0)}(f_1, f_2, \dots, f_m)(x)| \leq \left( \prod_{j: j \neq j', j''} M f_j(x) \right) S f_{j'}(x) S f_{j''}(x).$$

Now using generalized Hölder's inequality and the boundedness properties of the maximal and square functions, we get

$$\begin{aligned} \|P^{(0,0,\dots,0)}(f_1, f_2, \dots, f_m)\|_r &\leq \left( \prod_{j: j \neq j', j''} \|M f_j\|_{p_j} \right) \|S f_{j'}\|_{p_{j'}} \|S f_{j''}\|_{p_{j''}} \\ &\lesssim \prod_{j=1}^m \|f_j\|_{p_j}. \end{aligned}$$

□

**Lemma 3.4.** Let  $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in \{0, 1\}^m$  and  $1 < p_1, \dots, p_m, r < \infty$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ .

(a) For  $\sigma(\vec{\alpha}) \leq 1$ ,  $\pi_b^{\vec{\alpha}}$  is a bounded operator from  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^r$  if and only if  $b \in BMO^d$ .

(b) For  $\sigma(\vec{\alpha}) > 1$ ,  $\pi_b^{\vec{\alpha}}$  is a bounded operator from  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^r$  if and only if  $\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$ .

*Proof.* (a) We prove this part first for  $\sigma(\vec{\alpha}) = 0$ , that is, for  $\alpha_1 = \dots = \alpha_m = 1$ .

Assume that  $b \in BMO^d$ . Then for  $(f_1, \dots, f_m) \in L^{p_1} \times \dots \times L^{p_m}$ , we have

$$\begin{aligned}
\pi_b^{\vec{\alpha}}(f_1, \dots, f_m) &= P^{(0, \vec{\alpha})}(b, f_1, \dots, f_m) \\
&= \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \prod_{j=1}^m \langle f_j \rangle_I h_I \\
&= \sum_{I \in \mathcal{D}} \langle \pi_b(f_1), h_I \rangle \prod_{j=2}^m \langle f_j \rangle_I h_I \\
&= P^{(0, \alpha_2, \dots, \alpha_m)}(\pi_b(f_1), f_2, \dots, f_m).
\end{aligned}$$

Since  $b \in BMO^d$  and  $f_1 \in L^{p_1}$  with  $p_1 > 1$ , we have  $\|\pi_b(f_1)\|_{p_1} \lesssim \|b\|_{BMO^d} \|f_1\|_{p_1}$ . So,

$$\begin{aligned}
\|\pi_b^{\vec{\alpha}}(f_1, \dots, f_m)\|_r &= \|P^{(0, \alpha_2, \dots, \alpha_m)}(\pi_b(f_1), f_2, \dots, f_m)\|_r \\
&\lesssim \|\pi_b(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j},
\end{aligned}$$

where the first inequality follows from Theorem 3.3.

Conversely, assume that  $\pi_b^{(1, \dots, 1)} : L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r$  is bounded. Then for  $f_i = |J|^{-\frac{1}{p_i}} \mathbf{1}_J(x)$  with  $J \in \mathcal{D}$ ,

$$\left\| \pi_b^{(1, 1, \dots, 1)}(f_1, f_2, \dots, f_m) \right\|_r \leq \left\| \pi_b^{(1, 1, \dots, 1)} \right\|_{L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r},$$

since  $\|f_i\|_{p_i} = 1$  for all  $1 \leq i \leq m$ . For such  $f_i$ ,

$$\begin{aligned}
\left\| \pi_b^{(1, 1, \dots, 1)}(f_1, f_2, \dots, f_m) \right\|_r &= \left\| |J|^{-\left(\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}\right)} \pi_b^{(1, 1, \dots, 1)}(\mathbf{1}_J, \mathbf{1}_J, \dots, \mathbf{1}_J) \right\|_r \\
&= |J|^{-\frac{1}{r}} \left\| \sum_{I \in \mathcal{D}} \widehat{b}(I) \langle \mathbf{1}_J \rangle_I^m h_I \right\|_r.
\end{aligned}$$

Taking  $\epsilon_I = 1$  if  $I \subseteq J$  and  $\epsilon_I = 0$  otherwise, we observe that

$$\begin{aligned}
\left\| \sum_{J \supseteq I \in \mathcal{D}} \widehat{b}(I) h_I \right\|_r &= \left\| \sum_{J \supseteq I \in \mathcal{D}} \widehat{b}(I) \langle \mathbf{1}_J \rangle_I^m h_I \right\|_r \\
&= \left\| \sum_{I \in \mathcal{D}} \epsilon_I \widehat{b}(I) \langle \mathbf{1}_J \rangle_I^m h_I \right\|_r \\
&\lesssim \left\| \sum_{I \in \mathcal{D}} \widehat{b}(I) \langle \mathbf{1}_J \rangle_I^m h_I \right\|_r,
\end{aligned}$$

where the last inequality follows from the boundedness of Haar multiplier  $T_\epsilon$  on  $L^r$ . Thus, we have

$$\begin{aligned} \sup_{J \in \mathcal{D}} |J|^{-1/r} \left\| \sum_{J \supseteq I \in \mathcal{D}} \widehat{b}(I) h_I \right\|_r &\lesssim \sup_{J \in \mathcal{D}} |J|^{-1/r} \left\| \sum_{I \in \mathcal{D}} \widehat{b}(I) \langle \mathbf{1}_J \rangle_I^m h_I \right\|_r \\ &\lesssim \left\| \pi_b^{(1,1,\dots,1)} \right\|_{L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r}, \end{aligned}$$

proving that  $b \in BMO^d$ .

Now the proof for  $\sigma(\vec{\alpha}) = 1$  follows from the simple observation that  $\pi_b^{\vec{\alpha}}$  is a transpose of  $\pi_b^{(1,\dots,1)}$ . For example, if  $\sigma(\vec{\alpha}) = 1$  with  $\alpha_1 = 0$  and  $\alpha_2 = \dots = \alpha_m = 1$  and if  $r'$  is the conjugate exponent of  $r$ , then for  $g \in L^{r'}$

$$\begin{aligned} \langle \pi_b^{\vec{\alpha}}(f_1, \dots, f_m), g \rangle &= \left\langle \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f_1, h_I \rangle \prod_{j=2}^m \langle f_j \rangle_I h_I^2, g \right\rangle \\ &= \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f_1, h_I \rangle \prod_{j=2}^m \langle f_j \rangle_I \langle g, h_I^2 \rangle \\ &= \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f_1, h_I \rangle \prod_{j=1}^m \langle f_j \rangle_I \langle g \rangle_I \\ &= \left\langle \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle g \rangle_I \prod_{j=1}^m \langle f_j \rangle_I h_I, f_1 \right\rangle \\ &= \left\langle \pi_b^{(1,\dots,1)}(g, f_2, \dots, f_m), f_1 \right\rangle. \end{aligned}$$

(b) Assume that  $\|b\|_* \equiv \sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$ . For  $m = 2$  we have

$$\begin{aligned} \int_{\mathbb{R}} \left| \pi_b^{(0,0)}(f_1, f_2) \right|^r dx &= \int_{\mathbb{R}} \left| \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f_1, h_I \rangle \langle f_2, h_I \rangle h_I^3(x) \right|^r dx \\ &\leq \int_{\mathbb{R}} \left( \sum_{I \in \mathcal{D}} |\langle b, h_I \rangle| |\langle f_1, h_I \rangle| |\langle f_2, h_I \rangle| \frac{\mathbf{1}_I(x)}{|I|^{3/2}} \right)^r dx \\ &\leq \int_{\mathbb{R}} \left( \sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} \sum_{I \in \mathcal{D}} |\langle f_1, h_I \rangle| |\langle f_2, h_I \rangle| \frac{\mathbf{1}_I(x)}{|I|} \right)^r dx \\ &= \|b\|_*^r \int_{\mathbb{R}} \left( \sum_{I \in \mathcal{D}} |\langle f_1, h_I \rangle| |\langle f_2, h_I \rangle| \frac{\mathbf{1}_I(x)}{|I|} \right)^r dx. \end{aligned}$$

Using (3.3) and Hölder's inequality we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \left| \pi_b^{(0,0)}(f_1, f_2) \right|^r dx &\leq \|b\|_*^r \int_{\mathbb{R}} (Sf_1)^r(x) (Sf_2)^r(x) dx \\
&\leq \|b\|_*^r \left( \int_{\mathbb{R}} \{(Sf_1)^r(x)\}^{p_1/r} dx \right)^{r/p_1} \left( \int_{\mathbb{R}} \{(Sf_2)^r(x)\}^{p_2/r} dx \right)^{r/p_2} \\
&\leq \|b\|_*^r \|Sf_1\|_{p_1}^r \|Sf_2\|_{p_2}^r \\
&\lesssim \|b\|_*^r \|f_1\|_{p_1}^r \|f_2\|_{p_2}^r.
\end{aligned}$$

Thus we have,

$$\|\pi_b^{(0,0)}(f_1, f_2)\|_r \lesssim \|b\|_* \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Observe that

$$\pi_b^{(0,0)}(f_1, f_2)(I, 0) = \langle \pi_b^{(0,0)}(f_1, f_2), h_I \rangle = \frac{1}{|I|} \langle b, h_I \rangle \langle f_1, h_I \rangle \langle f_2, h_I \rangle.$$

Now consider  $m > 2$  and let  $\sigma(\vec{\alpha}) > 1$ . Without loss of generality we may assume that  $\alpha_1 = \alpha_2 = 0$ . Then

$$\begin{aligned}
\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r &= \left\| \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f_1, h_I \rangle \langle f_2, h_I \rangle \prod_{j=3}^m f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})} \right\|_r \\
&= \left\| \sum_{I \in \mathcal{D}} \frac{1}{|I|} \langle b, h_I \rangle \langle f_1, h_I \rangle \langle f_2, h_I \rangle \prod_{j=3}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})-1} \right\|_r \\
&= \left\| \sum_{I \in \mathcal{D}} \langle \pi_b^{(0,0)}(f_1, f_2), h_I \rangle \prod_{j=3}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})-1} \right\|_r \\
&= \left\| P^{\vec{\beta}}(\pi_b^{(0,0)}(f_1, f_2), f_3, \dots, f_m) \right\|_r \\
&\lesssim \|\pi_b^{(0,0)}(f_1, f_2)\|_q \prod_{j=3}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_* \prod_{j=1}^m \|f_j\|_{p_j}
\end{aligned}$$

where  $\vec{\beta} = (0, \alpha_3, \dots, \alpha_m) \in \{0, 1\}^{m-1}$  and  $\pi_b^{(0,0)}(f_1, f_2) \in L^q$  with  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q}$ ,  $q > r > 1$ .

Conversely, assume that  $\pi_b^{\vec{\alpha}} : L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r$  is bounded and that  $\sigma(\vec{\alpha}) > 1$ . Choose any  $J \in \mathcal{D}$ , and take  $f_j = |J|^{\frac{1}{2} - \frac{1}{p_j}} h_J$  if  $\alpha_j = 0$ , and  $f_j = |J|^{-\frac{1}{p_j}} 1_J$  if  $\alpha_j = 1$  so that  $\|f_j\|_{p_j} = 1$ . Then

$$\|\pi_b^{\vec{\alpha}}(f_1, \dots, f_m)\|_r \leq \|\pi_b^{\vec{\alpha}}\|_{L^{p_1} \times \dots \times L^{p_m}}.$$



We also have

$$\begin{aligned}
\|\pi_b^{\vec{\alpha}}(f_1, \dots, f_m)\|_r &= \left\| |J|^{\frac{\sigma(\vec{\alpha})}{2} - \sum_{j=1}^m \frac{1}{p_j}} \langle b, h_J \rangle h_J^{1+\sigma(\vec{\alpha})} \right\|_r \\
&= |J|^{\frac{\sigma(\vec{\alpha})}{2} - \frac{1}{r}} |\langle b, h_J \rangle| \|h_J^{1+\sigma(\vec{\alpha})}\|_r \\
&= |J|^{\frac{\sigma(\vec{\alpha})}{2} - \frac{1}{r}} |\langle b, h_J \rangle| |J|^{-\frac{1+\sigma(\vec{\alpha})}{2}} \|1_J\|_r \\
&= |J|^{\frac{\sigma(\vec{\alpha})}{2} - \frac{1}{r}} |\langle b, h_J \rangle| |J|^{-\frac{1+\sigma(\vec{\alpha})}{2}} |J|^{\frac{1}{r}} \\
&= \frac{|\langle b, h_J \rangle|}{\sqrt{|J|}}.
\end{aligned}$$

Thus  $\frac{|\langle b, h_J \rangle|}{\sqrt{|J|}} \leq \|\pi_b^{\vec{\alpha}}\|_{L^{p_1} \times \dots \times L^{p_m}}$ . Since it is true for any  $J \in D$ , we have

$$\sup_{J \in \mathcal{D}} \frac{|\langle b, h_J \rangle|}{\sqrt{|J|}} \leq \|\pi_b^{\vec{\alpha}}\|_{L^{p_1} \times \dots \times L^{p_m}} < \infty,$$

as desired. □

Now that we have obtained strong type  $L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r$  boundedness estimates for the paraproduct operators  $P^{\vec{\alpha}}$  and  $\pi_b^{\vec{\alpha}}$  when  $1 < p_1, p_2, \dots, p_m, r < \infty$  and  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ , we are interested to investigate estimates corresponding to  $\frac{1}{m} \leq r < \infty$ . We will prove in Lemma 3.6 that we obtain weak type estimates if one or more  $p_i$ 's are equal to 1. In particular, we obtain  $L^1 \times \dots \times L^1 \rightarrow L^{\frac{1}{m}, \infty}$  estimates for those operators. Then it follows from multilinear interpolation that the paraproduct operators are strongly bounded from  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^r$  for  $1 < p_1, p_2, \dots, p_m < \infty$  and  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ , even if  $\frac{1}{m} < r \leq 1$ .

We first prove the following general lemma, which when applied to the operators  $P^{\vec{\alpha}}$  and  $\pi_b^{\vec{\alpha}}$  gives aforementioned weak type estimates.

**Lemma 3.5.** *Let  $T$  be a multilinear operator that is bounded from the product of Lebesgue spaces  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^{r, \infty}$  for some  $1 < p_1, p_2, \dots, p_m < \infty$  with*

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

*Suppose that for every  $I \in \mathcal{D}$ ,  $T(f_1, \dots, f_m)$  is supported in  $I$  if  $f_i = h_I$  for some  $i \in \{1, 2, \dots, m\}$ . Then  $T$  is bounded from  $L^1 \times \dots \times L^1 \times L^{p_{k+1}} \times \dots \times L^{p_m} \rightarrow L^{\frac{q_k}{q_k+1}, \infty}$  for each  $k = 1, 2, \dots, m$ , where  $q_k$  is given by*

$$\frac{1}{q_k} = (k-1) + \frac{1}{p_{k+1}} + \dots + \frac{1}{p_m}.$$

*In particular,  $T$  is bounded from  $L^1 \times \dots \times L^1$  to  $L^{\frac{1}{m}, \infty}$ .*

*Proof.* We first prove that  $T$  is bounded from  $L^1 \times L^{p_2} \times \cdots \times L^{p_m}$  to  $L^{\frac{q_1}{q_1+1}, \infty}$ .  
Let  $\lambda > 0$  be given. We have to show that

$$|\{x : |T(f_1, f_2, \dots, f_m)(x)| > \lambda\}| \lesssim \left( \frac{\|f_1\|_1 \prod_{j=2}^m \|f_j\|_{p_j}}{\lambda} \right)^{\frac{q_1}{1+q_1}}$$

for all  $(f_1, f_2, \dots, f_m) \in L^1 \times L^{p_2} \times \cdots \times L^{p_m}$ .

Without loss of generality, we assume  $\|f_1\|_1 = \|f_2\|_{p_2} = \cdots = \|f_m\|_{p_m} = 1$ , and prove that

$$|\{x : |T(f_1, f_2, \dots, f_m)(x)| > \lambda\}| \lesssim \lambda^{-\frac{q_1}{1+q_1}}.$$

For this, we apply Calderón-Zygmund decomposition to the function  $f_1$  at height  $\lambda^{\frac{q_1}{q_1+1}}$  to obtain ‘good’ and ‘bad’ functions  $g_1$  and  $b_1$ , and a sequence  $\{I_{1,j}\}$  of disjoint dyadic intervals such that

$$f_1 = g_1 + b_1, \quad \|g_1\|_{p_1} \leq \left(2\lambda^{\frac{q_1}{q_1+1}}\right)' \|f_1\|_1^{1/p_1} = \left(2\lambda^{\frac{q_1}{q_1+1}}\right)^{\frac{p_1-1}{p_1}} \quad \text{and} \quad b_1 = \sum_j b_{1,j},$$

where

$$\text{supp}(b_{1,j}) \subseteq I_{1,j}, \quad \int_{I_{1,j}} b_{1,j} dx = 0, \quad \text{and} \quad \sum_j |I_{1,j}| \leq \lambda^{-\frac{q_1}{q_1+1}} \|f_1\|_1 = \lambda^{-\frac{q_1}{q_1+1}}.$$

Multilinearity of  $T$  implies that

$$\begin{aligned} & |\{x : |T(f_1, \dots, f_m)(x)| > \lambda\}| \\ & \leq \left| \left\{ x : |T(g_1, f_2, \dots, f_m)(x)| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ x : |T(b_1, f_2, \dots, f_m)(x)| > \frac{\lambda}{2} \right\} \right|. \end{aligned}$$

Since  $g_1 \in L^{p_1}$  and  $T$  is bounded from  $L^{p_1} \times \cdots \times L^{p_m}$  to  $L^{r, \infty}$ , we have

$$\begin{aligned} |\{x : |T(g_1, f_2, \dots, f_m)(x)| > \lambda/2\}| & \lesssim \left( \frac{2\|g_1\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j}}{\lambda} \right)^r \\ & \leq \left( \frac{2 \left(2\lambda^{\frac{q_1}{q_1+1}}\right)^{\frac{p_1-1}{p_1}}}{\lambda} \right)^r \\ & \lesssim \lambda^{r \left( \frac{q_1(p_1-1)}{p_1(q_1+1)} - 1 \right)} \end{aligned}$$

Now,  $\frac{1}{r} = \sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p_1} + \frac{1}{q_1}$  implies that  $r = \frac{p_1 q_1}{p_1 + q_1}$ . So,

$$\begin{aligned} r \left( \frac{q_1(p_1-1)}{p_1(q_1+1)} - 1 \right) &= \frac{p_1 q_1}{(p_1 + q_1)} \left( \frac{p_1 q_1 - q_1 - p_1 q_1 - p_1}{p_1(q_1+1)} \right) \\ &= \frac{p_1 q_1}{(p_1 + q_1)} \frac{(-p_1 - q_1)}{p_1(q_1+1)} \\ &= -\frac{q_1}{q_1 + 1}. \end{aligned}$$

Thus we have:

$$|\{x : |T(g_1, f_2, \dots, f_m)(x)| > \lambda/2\}| \lesssim \lambda^{-\frac{q_1}{1+q_1}}.$$

From the properties of ‘bad’ function  $b_1$  we deduce that  $\langle b_1, h_I \rangle \neq 0$  only if  $I \subseteq I_{1,j}$  for some  $j$ . The hypothesis of the lemma on the support of  $T(f_1, \dots, f_m)$  then implies that

$$\text{supp}(T(b_1, f_2, \dots, f_m)) \subseteq \cup_j I_{1,j}.$$

Thus,

$$\left| \left\{ x : |T(b_1, f_2, \dots, f_m)(x)| > \frac{\lambda}{2} \right\} \right| \leq |\cup_j I_{1,j}| \leq \lambda^{-\frac{q_1}{1+q_1}}.$$

Combining these estimates corresponding to  $g_1$  and  $b_1$ , we have the desired estimate

$$|\{x : |T(f_1, f_2, \dots, f_m)(x)| > \lambda\}| \lesssim \lambda^{-\frac{q_1}{1+q_1}}.$$

Now beginning with the  $L^1 \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^{\frac{q_1}{q_1+1}, \infty}$  estimate, we use the same argument to lower the second exponent to 1 proving that  $T$  is bounded from  $L^1 \times L^1 \times L^{p_3} \times \dots \times L^{p_m}$  to  $L^{\frac{q_2}{q_2+1}, \infty}$ , where  $q_2$  is given by  $\frac{1}{q_2} = 1 + \frac{1}{p_3} + \dots + \frac{1}{p_m}$ .

We continue the same process until we obtain  $L^1 \times L^1 \times \dots \times L^1 \rightarrow L^{\frac{q_m}{q_m+1}, \infty}$  boundedness of  $T$  with  $\frac{1}{q_m} = 1 + 1 + \dots + 1$  ( $m-1$  terms)  $= m-1$ . This completes the proof since  $\frac{q_m}{q_m+1} = \frac{1}{m}$ .  $\square$

**Lemma 3.6.** *Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$ ,  $1 \leq p_1, p_2, \dots, p_m < \infty$  and  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ . Then*

- (a) *For  $\vec{\alpha} \neq (1, 1, \dots, 1)$ ,  $P^{\vec{\alpha}}$  is bounded from  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^{r, \infty}$ .*
- (b) *If  $b \in BMO^d$  and  $\sigma(\vec{\alpha}) \leq 1$ ,  $\pi_b^{\vec{\alpha}}$  is bounded from  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^{r, \infty}$ .*
- (c) *If  $\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$  and  $\sigma(\vec{\alpha}) > 1$ ,  $\pi_b^{\vec{\alpha}}$  is bounded from  $L^{p_1} \times \dots \times L^{p_m}$  to  $L^{r, \infty}$ .*

*Proof.* By orthogonality of Haar functions,  $h_I(J, 0) = \langle h_I, h_J \rangle = 0$  for any two distinct dyadic intervals  $I$  and  $J$ . The Haar functions have mean value 0, so it is easy to see that

$$\langle h_I \rangle_J \neq 0 \text{ only if } J \subsetneq I$$

since any two dyadic intervals are either disjoint or one is contained in the other.

Consequently, if some  $f_i = h_I$ , then

$$P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{J \subseteq I} \prod_{j=1}^m f_j(J, \alpha_j) h_J^{\sigma(\vec{\alpha})}$$

and,

$$\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{J \subseteq I} \langle b, h_J \rangle \prod_{j=1}^m f_j(J, \alpha_j) h_J^{1+\sigma(\vec{\alpha})},$$

which are both supported in  $I$ . Since the paraproducts are strongly (and hence weakly) bounded from  $L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r$ , the proof follows immediately from Lemma 3.5.  $\square$

Combining the results of Lemmas 3.3, 3.4 and 3.6, and using multilinear interpolation (see [4]), we have the following theorem:

**Theorem 3.7.** Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$  and  $1 < p_1, p_2, \dots, p_m < \infty$  with  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$ . Then

- (a) For  $\vec{\alpha} \neq (1, 1, \dots, 1)$ ,  $\|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \lesssim \prod_{j=1}^m \|f_j\|_{p_j}$ .
- (b) For  $\sigma(\vec{\alpha}) \leq 1$ ,  $\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}$ , if and only if  $b \in BMO^d$ .
- (c) For  $\sigma(\vec{\alpha}) > 1$ ,  $\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \leq C_b \prod_{j=1}^m \|f_j\|_{p_j}$ , if and only if  $\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$ .

In each of the above cases, the paraproducts are weakly bounded if  $1 \leq p_1, p_2, \dots, p_m < \infty$ .

#### 4. MULTILINEAR HAAR MULTIPLIERS AND MULTILINEAR COMMUTATORS

**4.1. Multilinear Haar Multipliers.** In this subsection we introduce multilinear Haar multipliers, and study their boundedness properties.

**Definition 4.1.** Given  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$ , and a symbol sequence  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$ , we define  $m$ -linear Haar multipliers by

$$T_{\epsilon}^{\vec{\alpha}}(f_1, f_2, \dots, f_m) \equiv \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}.$$

**Theorem 4.1.** Let  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$  be a given sequence and let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ . Let  $1 < p_1, p_2, \dots, p_m < \infty$  with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

Then  $T_{\epsilon}^{\vec{\alpha}}$  is bounded from  $L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}$  to  $L^r$  if and only if  $\|\epsilon\|_{\infty} := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$ .

Moreover,  $T_{\epsilon}^{\vec{\alpha}}$  has the corresponding weak-type boundedness if  $1 \leq p_1, p_2, \dots, p_m < \infty$ .

*Proof.* To prove this lemma we use the fact that the linear Haar multiplier

$$T_{\epsilon}(f) = \sum_{I \in \mathcal{D}} \epsilon_I \langle f, h_I \rangle h_I$$

is bounded on  $L^p$  for all  $1 < p < \infty$  if  $\|\epsilon\|_{\infty} := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$ , and that  $\langle T_{\epsilon}(f), h_I \rangle = \epsilon_I \langle f, h_I \rangle$ .

By assumption  $\sigma(\vec{\alpha}) \geq 1$ . Without loss of generality we may assume that  $\alpha_i = 0$  if  $1 \leq i \leq \sigma(\vec{\alpha})$  and  $\alpha_i = 1$  if  $\sigma(\vec{\alpha}) < i \leq m$ . In particular, we have  $\alpha_1 = 0$ . Then

$$\epsilon_I f_1(I, \alpha_1) = \epsilon_I \langle f_1, h_I \rangle = \langle T_{\epsilon}(f_1), h_I \rangle = T_{\epsilon}(f_1)(I, \alpha_1).$$

First assume that  $\|\epsilon\|_{\infty} := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$ .

Then,

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r &= \left\| \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right\|_r \\
&= \left\| \sum_{I \in \mathcal{D}} T_\epsilon(f_1)(I, \alpha_1) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right\|_r \\
&= \|P^{\vec{\alpha}}(T_\epsilon(f_1), f_2, \dots, f_m)\|_r \\
&\lesssim \|T_\epsilon(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \prod_{j=1}^m \|f_j\|_{p_j}.
\end{aligned}$$

Conversely, assume that  $T_\epsilon^{\vec{\alpha}} : L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^r$  is bounded, and let  $\sigma(\vec{\alpha}) = k$ . Recall that  $\alpha_i = 0$  if  $1 \leq i \leq \sigma(\vec{\alpha}) = k$  and  $\alpha_i = 1$  if  $k = \sigma(\vec{\alpha}) < i \leq m$ . Taking  $f_i = h_I$  if  $1 \leq i \leq k$  and  $f_i = 1_I$  if  $k < i \leq m$ , we observe that

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r &= \left( \int_{\mathbb{R}} |\epsilon_I h_I^k(x)|^r dx \right)^{1/r} \\
&= \left( \frac{|\epsilon_I|^r}{|I|^{kr/2}} \int_{\mathbb{R}} 1_I(x) dx \right)^{1/r} \\
&= \frac{|\epsilon_I|}{|I|^{k/2}} |I|^{1/r}
\end{aligned}$$

and,

$$\begin{aligned}
\prod_{j=1}^m \|f_j\|_{p_j} &= \prod_{i=1}^k \left( \int_{\mathbb{R}} |h_I(x)|^{p_i} dx \right)^{1/p_i} \prod_{j=k+1}^m \left( \int_{\mathbb{R}} |1_I(x)|^{p_j} dx \right)^{1/p_j} \\
&= \prod_{i=1}^k \left( \frac{1}{|I|^{p_i/2}} \int_{\mathbb{R}} 1_I(x) dx \right)^{1/p_i} \prod_{j=k+1}^m \left( \int_{\mathbb{R}} 1_I(x) dx \right)^{1/p_j} \\
&= \prod_{i=1}^k \left( \frac{1}{|I|^{1/2}} |I|^{1/p_i} \right) \prod_{j=k+1}^m |I|^{1/p_j} \\
&= \frac{|I|^{1/r}}{|I|^{k/2}}
\end{aligned}$$

Since  $(f_1, f_2, \dots, f_m) \in L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}$ , the boundedness of  $T_\epsilon$  implies that

$$\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \leq \|T_\epsilon^{\vec{\alpha}}\|_{L^{p_1} \times \dots \times L^{p_m} \rightarrow L^r} \prod_{j=1}^m \|f_j\|_{p_j}.$$

That is,

$$\frac{|\epsilon_I|}{|I|^{k/2}} |I|^{1/r} \leq \|T_\epsilon^{\vec{\alpha}}\|_{L^{p_1} \times \dots \times L^{p_m}} \frac{|I|^{1/r}}{|I|^{k/2}},$$

for all  $I \in \mathcal{D}$ . Consequently,  $\|\epsilon\|_\infty = \sup_{I \in \mathcal{D}} |\epsilon_I| \leq \|T_\epsilon^{\vec{\alpha}}\|_{L^{p_1} \times \dots \times L^{p_m}} < \infty$ , as desired.

If  $1 \leq p_1, p_2, \dots, p_m < \infty$ , the weak-type boundedness of  $T_\epsilon^{\vec{\alpha}}$  follows from Lemma 3.5.  $\square$

**4.2. Multilinear commutators.** In this subsection we study boundedness properties of the commutators of  $T_\epsilon^{\vec{\alpha}}$  with the multiplication operator  $M_b$  when  $b \in BMO^d$ . For convenience we denote the operator  $M_b$  by  $b$  itself. We are interested in the following commutators:

$$[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)(x) \equiv (T_\epsilon^{\vec{\alpha}}(f_1, \dots, bf_i, \dots, f_m) - bT_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m))(x)$$

where  $1 \leq i \leq m$ .

Note that if  $b$  is a constant function,  $[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)(x) = 0$  for all  $x$ . Our approach to study the boundedness properties of  $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^r$  with

$1 < p_1, p_2, \dots, p_m < \infty$  and  $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}$  for non-constant  $b$  requires us to assume that  $b \in L^p$

for some  $p \in (1, \infty)$ , and that  $r > 1$ . However, this restricted unweighted theory turns out to be sufficient to obtain a weighted theory, which in turn implies the unrestricted unweighted theory of these multilinear commutators. We will present the weighted theory of these commutators in a subsequent paper.

**Theorem 4.2.** *Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ . If  $b \in BMO^d \cap L^p$  for some  $1 < p < \infty$  and  $\|\epsilon\|_\infty := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$ , then each commutator  $[b, T_\epsilon^{\vec{\alpha}}]_i$  is bounded from  $L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^r$  for all  $1 < p_1, p_2, \dots, p_m, r < \infty$  with*

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r},$$

with estimates of the form:

$$\|[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.$$

*Proof.* It suffices to prove boundedness of  $[b, T_\epsilon^{\vec{\alpha}}]_1$ , as the others are identical. Moreover, we may assume that each  $f_i$  is bounded and has compact support, since such functions are dense in the  $L^p$  spaces.

Writing  $bf_1 = \pi_b(f_1) + \pi_b^*(f_1) + \pi_{f_1}(b)$  and using multilinearity of  $T_\epsilon^{\vec{\alpha}}$ , we have

$$T_\epsilon^{\vec{\alpha}}(bf_1, f_2, \dots, f_m) = T_\epsilon^{\vec{\alpha}}(\pi_b(f_1), f_2, \dots, f_m) + T_\epsilon^{\vec{\alpha}}(\pi_b^*(f_1), f_2, \dots, f_m) + T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m).$$

On the other hand,

$$\begin{aligned}
bT_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) &= \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \left( \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \right) \\
&= \sum_{I \in \mathcal{D}} \epsilon_I \widehat{b}(I) \prod_{j=1}^m f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})} \\
&\quad + \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \left( \sum_{I \subsetneq J} \widehat{b}(J) h_J \right) \\
&\quad + \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \left( \sum_{J \subsetneq I} \widehat{b}(J) h_J \right) \\
&= \pi_b^{\vec{\alpha}}(f_1, \dots, T_\epsilon(f_i), \dots, f_m) \\
&\quad + \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&\quad + \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \left( \sum_{J \subsetneq I} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right)
\end{aligned}$$

for some  $i$  with  $\alpha_i = 0$ . Indeed, some  $\alpha_i$  equals 0 by assumption, and for such  $i$ , we have

$$T_\epsilon(f_i)(I, \alpha_i) = \widehat{T_\epsilon(f_i)}(I) = \epsilon_I \widehat{f_i}(I) = \epsilon_I f_i(I, \alpha_i).$$

For  $(f_1, f_2, \dots, f_m) \in L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}$ , we have

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(\pi_b(f_1), f_2, \dots, f_m)\|_r &\lesssim \|\pi_b(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}
\end{aligned}$$

$$\begin{aligned}
\|T_\epsilon^{\vec{\alpha}}(\pi_b^*(f_1), f_2, \dots, f_m)\|_r &\lesssim \|\pi_b^*(f_1)\|_{p_1} \prod_{j=2}^m \|f_j\|_{p_j} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.
\end{aligned}$$

and,

$$\begin{aligned}
\|\pi_b^{\vec{\alpha}}(f_1, \dots, T_\epsilon(f_i), \dots, f_m)\|_r &\lesssim \|b\|_{BMO^d} \|f_1\|_{p_1} \cdots \|T_\epsilon(f_i)\|_{p_i} \cdots \|f_m\|_{p_m} \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.
\end{aligned}$$

So, to prove boundedness of  $[b, T_\epsilon^{\vec{\alpha}}]_1$ , it suffices to show similar control over the terms:

$$(4.1) \quad \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \left( \sum_{J \subsetneq I} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \right\|_r$$

and,

$$(4.2) \quad \left\| T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m) - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right\|_r.$$

**Estimation of (4.1):**

Case I:  $\sigma(\vec{\alpha})$  odd.

In this case,

$$T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} = \sum_{I \in \mathcal{D}} \epsilon_I |I|^{\frac{1-\sigma(\vec{\alpha})}{2}} \prod_{j=1}^m f_j(I, \alpha_j) h_I.$$

So,

$$\langle T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m), h_I \rangle h_I = \epsilon_I |I|^{\frac{1-\sigma(\vec{\alpha})}{2}} \prod_{j=1}^m f_j(I, \alpha_j) h_I = \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}.$$

This implies that

$$\begin{aligned} (4.1) &= \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \left( \sum_{J \subsetneq I} \langle T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m), h_I \rangle h_I \right) \right\|_r \\ &= \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \langle T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) \rangle_J h_J \right\|_r \\ &= \left\| \pi_b(T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)) \right\|_r \\ &\lesssim \|b\|_{BMO^d} \|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_r \\ &\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}. \end{aligned}$$

Case II:  $\sigma(\vec{\alpha})$  even.

In this case at least two  $\alpha'_i$ 's are equal to 0. Without loss of generality we may assume that  $\alpha_1 = 0$ . Then denoting  $T_\epsilon(f_1)$  by  $g_1$ ,  $P^{(\alpha_2, \dots, \alpha_m)}(f_2, \dots, f_m)$  by  $g_2$ , and using the fact that

$$\langle g_1 \rangle_J \langle g_2 \rangle_J \mathbf{1}_J = \left( \sum_{J \subsetneq I} \widehat{g}_1(I) \langle g_2 \rangle_I h_I + \sum_{J \subsetneq I} \langle g_1 \rangle_I \widehat{g}_2(I) h_I + \sum_{J \subsetneq I} \widehat{g}_1(I) \widehat{g}_2(I) h_I^2 \right) \mathbf{1}_J,$$



we have

$$\begin{aligned}
& \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \left( \sum_{J \subsetneq I} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \right\|_r \\
&= \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \left( \sum_{J \subsetneq I} \widehat{g}_1(I) \widehat{g}_2(I) h_I^2 \right) \right\|_r \\
&= \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) h_J \left( \langle g_1 \rangle_J \langle g_2 \rangle_J 1_J - \sum_{J \subsetneq I} \widehat{g}_1(I) \langle g_2 \rangle_I h_I - \sum_{J \subsetneq I} \langle g_1 \rangle_I \widehat{g}_2(I) h_I \right) \right\|_r \\
&\leq \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \langle g_1 \rangle_J \langle g_2 \rangle_J h_J \right\|_r + \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \langle P^{(0,1)}(g_1, g_2) \rangle_J h_J \right\|_r \\
&\quad + \left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \langle P^{(1,0)}(g_1, g_2) \rangle_J h_J \right\|_r \\
&\lesssim \|b\|_{BMO^d} \|g_1\|_{p_1} \|g_2\|_q + \|b\|_{BMO^d} \|P^{(0,1)}(g_1, g_2)\|_r + \|b\|_{BMO^d} \|P^{(1,0)}(g_1, g_2)\|_r \\
&\lesssim \|b\|_{BMO^d} \|g_1\|_{p_1} \|g_2\|_q \\
&\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.
\end{aligned}$$

where,  $q$  is given by  $\frac{1}{q} = \sum_{j=2}^m \frac{1}{p_j}$ . Here the last three inequalities follow from Theorems 3.3 and 3.4, and the fact that  $\|g_1\|_{p_1} = \|T_\epsilon(f_1)\|_{p_1} \lesssim \|f_1\|_{p_1}$ .

**Estimation of (4.2) :**

Case I:  $\alpha_1 = 0$ .

This case is easy as we observe that

$$\begin{aligned}
& T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m) - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= \sum_{I \in \mathcal{D}} \epsilon_I \widehat{\pi_{f_1}(b)}(I) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \widehat{f_1}(I) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \widehat{f_1}(I) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \widehat{f_1}(I) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= 0.
\end{aligned}$$

So there is nothing to estimate.

Case II:  $\alpha_1 = 1$ .

In this case,

$$\begin{aligned}
& T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m) - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= \sum_{I \in \mathcal{D}} \epsilon_I \langle \pi_{f_1}(b) \rangle_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \langle f_1 \rangle_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= \sum_{I \in \mathcal{D}} \epsilon_I (\langle \pi_{f_1}(b) \rangle_I - \langle b \rangle_I \langle f_1 \rangle_I) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}
\end{aligned}$$

Now,

$$\begin{aligned}
\langle b \rangle_I \langle f_1 \rangle_I 1_I &= \sum_{I \subsetneq J} \widehat{b}(J) \langle f_1 \rangle_J h_J 1_I + \sum_{I \subsetneq J} \langle b \rangle_J \widehat{f_1}(J) h_J 1_I + \sum_{I \subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 1_I \\
&= \langle \pi_b(f_1) \rangle_I 1_I + \langle \pi_{f_1}(b) \rangle_I 1_I + \sum_{I \subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 1_I.
\end{aligned}$$

$$\text{Hence, } \langle b \rangle_I \langle f_1 \rangle_I 1_I - \langle \pi_{f_1}(b) \rangle_I 1_I = \langle \pi_b(f_1) \rangle_I 1_I + \sum_{I \subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 1_I.$$

So we have

$$\begin{aligned}
& T_\epsilon^{\vec{\alpha}}(\pi_{f_1}(b), f_2, \dots, f_m) - \sum_{I \in \mathcal{D}} \epsilon_I \langle b \rangle_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= - \sum_{I \in \mathcal{D}} \epsilon_I \left( \langle \pi_b(f_1) \rangle_I 1_I + \sum_{I \subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \right) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= - \sum_{I \in \mathcal{D}} \epsilon_I \langle \pi_b(f_1) \rangle_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&\quad - \sum_{I \in \mathcal{D}} \epsilon_I \left( \sum_{I \subsetneq J} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \right) \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \\
&= -T_\epsilon(\pi_b(f_1), f_2, \dots, f_m) - \sum_{J \in \mathcal{D}} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \left( \sum_{I \subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right).
\end{aligned}$$

Since

$$\|T_\epsilon(\pi_b(f_1), f_2, \dots, f_m)\|_r \lesssim \|\pi_b(f_1)\|_{p_1} \prod_{j=2}^m f_j(J, \alpha_j) \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j},$$

we are left with controlling

$$\left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \widehat{f_1}(J) h_J^2 \left( \sum_{I \subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \right\|_r.$$

For this we observe that

$$\|T_\epsilon^{(\alpha_2, \dots, \alpha_m)}(f_2, \dots, f_m)\|_q \lesssim \prod_{j=2}^m \|f_j\|_{p_j},$$

and that

$$\begin{aligned} \pi_b^*(f_1) T_\epsilon^{(\alpha_2, \dots, \alpha_m)}(f_2, \dots, f_m) &= \sum_{J \in \mathcal{D}} \widehat{b}(J) \widehat{f}_1(J) h_J^2 \left( \sum_{I \subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \\ &\quad + \sum_{J \in \mathcal{D}} \epsilon_J \widehat{b}(J) \widehat{f}_1(J) \prod_{j=2}^m f_j(J, \alpha_j) h_J^{2+\sigma(\vec{\alpha})} \\ &\quad + \sum_{J \in \mathcal{D}} \widehat{b}(J) \widehat{f}_1(J) h_J^2 \left( \sum_{J \subsetneq I} \epsilon_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \end{aligned}$$

Now, following the same technique we used to control (4.1), we obtain

$$\left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \widehat{f}_1(J) h_J^2 \left( \sum_{J \subsetneq I} \epsilon_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \right\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.$$

We also have

$$\begin{aligned} \|\pi_b^*(f_1) T_\epsilon^{(\alpha_2, \dots, \alpha_m)}(f_2, \dots, f_m)\|_r &\leq \|\pi_b^*(f_1)\|_{p_1} \|T_\epsilon^{(\alpha_2, \dots, \alpha_m)}(f_2, \dots, f_m)\|_q \\ &\lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j} \end{aligned}$$

and,

$$\left\| \sum_{J \in \mathcal{D}} \epsilon_J \widehat{b}(J) \widehat{f}_1(J) \prod_{j=2}^m f_j(J, \alpha_j) h_J^{2+\sigma(\vec{\alpha})} \right\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.$$

So we conclude that

$$\left\| \sum_{J \in \mathcal{D}} \widehat{b}(J) \widehat{f}_1(J) h_J^2 \left( \sum_{I \subsetneq J} \epsilon_I \prod_{j=2}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})} \right) \right\|_r \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.$$

Thus we have strong type boundedness of

$$[b, T_\epsilon^{\vec{\alpha}}]_1 \rightarrow L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^r$$

for all  $1 < p_1, p_2, \dots, p_m, r < \infty$  with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

□

In the next theorem, we show that BMO condition is necessary for the boundedness of the commutators.

**Theorem 4.3.** Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ , and  $1 < p_1, p_2, \dots, p_m, r < \infty$  with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

Assume that for given  $b$  and  $i$ ,

$$(4.3) \quad \|[b, T_\epsilon^\alpha]_i(f_1, f_2, \dots, f_m)\|_r \leq C_\epsilon \prod_{j=1}^m \|f_j\|_{p_j},$$

for every bounded sequence  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$ , and for all  $f_i \in L^{p_i}$ . Then  $b \in BMO^d$ .

*Proof.* Without loss of generality we may assume that  $i = 1$ . Fix  $I_0 \in \mathcal{D}$  and let  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$  with  $\epsilon_I = 1$  for all  $I \in \mathcal{D}$ .

**Case I:**  $\alpha_1 = 0, \sigma(\vec{\alpha}) = 1$ .

Take  $f_1 = 1_{I_0}$  and  $f_i = h_{I_0^{(1)}}$  for  $i > 1$ , where  $I_0^{(1)}$  is the parent of  $I_0$ . Then,

$$T_\epsilon^\alpha(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \langle 1_{I_0}, h_I \rangle \langle h_{I_0^{(1)}} \rangle_I^{m-1} h_I = 0,$$

and,

$$\begin{aligned} T_\epsilon^\alpha(bf_1, f_2, \dots, f_m) &= \sum_{I \in \mathcal{D}} \langle b1_{I_0}, h_I \rangle \langle h_{I_0^{(1)}} \rangle_I^{m-1} h_I \\ &= \sum_{I \subseteq I_0} \langle b1_{I_0}, h_I \rangle \left( \frac{K(I_0, I_0^{(1)})}{\sqrt{|I_0^{(1)}|}} \right)^{m-1} h_I \\ &= \left( \frac{K(I_0, I_0^{(1)})}{\sqrt{|I_0^{(1)}|}} \right)^{m-1} \sum_{I \subseteq I_0} \langle b, h_I \rangle h_I, \end{aligned}$$

where  $K(I_0, I_0^{(1)})$  is either 1 or  $-1$  depending on whether  $I_0$  is the right or left half of  $I_0^{(1)}$ . For the second to last equality we observe that, if  $I$  is not a proper subset of  $I_0^{(1)}$ ,  $\langle h_{I_0^{(1)}} \rangle_I = 0$ , and that if  $I$  is a proper subset of  $I_0^{(1)}$  but is not a subset of  $I_0$ , then  $\langle b1_{I_0}, h_I \rangle = 0$ . Moreover, for  $I \subseteq I_0$ ,  $\langle b1_{I_0}, h_I \rangle = \int_{\mathbb{R}} b1_{I_0} h_I = \int_{\mathbb{R}} b h_I = \langle b, h_I \rangle$ .

Now from inequality (4.3), we get

$$\left\| \left( \frac{K(I_0, I_0^{(1)})}{\sqrt{|I_0^{(1)}|}} \right)^{m-1} \sum_{I \subseteq I_0} \langle b, h_I \rangle h_I \right\|_r \leq C_\epsilon |I_0|^{\frac{1}{p_1}} \prod_{i=2}^m \frac{|I_0^{(1)}|^{\frac{1}{p_i}}}{\sqrt{|I_0^{(1)}|}}$$

$$\text{i.e.} \quad \left\| \sum_{I \subseteq I_0} \langle b, h_I \rangle h_I \right\|_r \leq 2^{\frac{1}{p_2} + \dots + \frac{1}{p_m}} C_\epsilon |I_0|^{\frac{1}{r}}.$$

Thus for every  $I_0 \in \mathcal{D}$ ,

$$\frac{1}{|I_0|^{\frac{1}{r}}} \left\| \sum_{I \subseteq I_0} \langle b, h_I \rangle h_I \right\|_r \leq 2^{\frac{1}{p_2} + \dots + \frac{1}{p_m}} C_\epsilon,$$

and hence  $b \in BMO^d$ .

**Case II:**  $\alpha_1 \neq 0$  or  $\sigma(\vec{\alpha}) > 1$ .

Taking  $f_i = \begin{cases} h_{I_0}, & \text{if } \alpha_i = 0 \\ 1_{I_0}, & \text{if } \alpha_i = 1, \end{cases}$  we observe that

$$T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = h_{I_0}^{\sigma(\vec{\alpha})} \quad \text{and} \quad T_\epsilon^{\vec{\alpha}}(bf_1, f_2, \dots, f_m) = (bf_1)(I_0, \alpha_1) h_{I_0}^{\sigma(\vec{\alpha})}.$$

If  $\alpha_1 = 0$ ,

$$(bf_1)(I_0, \alpha_1) = bh_{I_0}(I_0, 0) = \widehat{bh_{I_0}}(I_0) = \int_{\mathbb{R}} bh_{I_0} h_{I_0} = \frac{1}{|I_0|} \int_{\mathbb{R}} b 1_{I_0} = \langle b \rangle_{I_0}.$$

If  $\alpha_1 = 1$ ,

$$(bf_1)(I_0, \alpha_1) = b 1_{I_0}(I_0, 1) = \langle b 1_{I_0} \rangle_{I_0} = \langle b \rangle_{I_0}.$$

So in each case,

$$\begin{aligned} \| [b, T_\epsilon^{\vec{\alpha}}]_1(f_1, f_2, \dots, f_m) \|_r &= \| bT_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) - T_\epsilon^{\vec{\alpha}}(bf_1, f_2, \dots, f_m) \|_r \\ &= \| bh_{I_0}^{\sigma(\vec{\alpha})} - \langle b \rangle_{I_0} h_{I_0}^{\sigma(\vec{\alpha})} \|_r \\ &= \| (b - \langle b \rangle_{I_0}) h_{I_0}^{\sigma(\vec{\alpha})} \|_r \\ &= \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} \| (b - \langle b \rangle_{I_0}) 1_{I_0} \|_r. \end{aligned}$$

On the other hand,

$$\prod_{j=1}^m \| f_j \|_{p_j} = \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} |I_0|^{\frac{1}{p_1} + \dots + \frac{1}{p_m}} = \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} |I_0|^{\frac{1}{r}}.$$

Inequality (4.3) then gives

$$\begin{aligned} \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} \| (b - \langle b \rangle_{I_0}) 1_{I_0} \|_r &\leq C_\epsilon \frac{1}{(\sqrt{|I_0|})^{\sigma(\vec{\alpha})}} |I_0|^{\frac{1}{r}} \\ \text{i.e.} \quad \frac{1}{|I_0|^{\frac{1}{r}}} \| (b - \langle b \rangle_{I_0}) 1_{I_0} \|_r &\leq C_\epsilon. \end{aligned}$$

Since this is true for any  $I_0 \in \mathcal{D}$ , we have  $b \in BMO^d$ . □

Combining the results from Theorems 4.2 and 4.3, we have the following characterization of the dyadic BMO functions. Note that if  $\epsilon_I = 1$  for every  $I \in \mathcal{D}$ , we have  $T_\epsilon^{\vec{\alpha}} = P^{\vec{\alpha}}$ , and that in the proof of Theorem 4.3, only the boundedness of  $[b, T_\epsilon^{\vec{\alpha}}]_i$  for  $\epsilon$  with  $\epsilon_I = 1$  for all  $I \in \mathcal{D}$  was used to show that  $b \in BMO^d$ .

**Theorem 4.4.** Let  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$ ,  $1 \leq i \leq m$ , and  $1 < p_1, p_2, \dots, p_m, r < \infty$  with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{r}.$$

Suppose  $b \in L^p$  for some  $p \in (1, \infty)$ . Then the following two statements are equivalent.

(a)  $b \in BMO^d$ .

(b)  $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^r$  is bounded for every bounded sequence  $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$ .

In particular,  $b \in BMO^d$  if and only if  $[b, P^{\vec{\alpha}}]_i : L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow L^r$  is bounded.

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ISHWARI KUNWAR, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, 686 CHERRY STREET, ATLANTA, GA USA 30332-0160

E-mail address: [ikunwar3@math.gatech.edu](mailto:ikunwar3@math.gatech.edu)